

# ROMAN $\{2\}$ -DOMINATION IN GRAPHS AND GRAPH PRODUCTS

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**ABSTRACT.** For a graph  $G = (V, E)$  of order  $n$ , a Roman  $\{2\}$ -dominating function  $f : V \rightarrow \{0, 1, 2\}$  has the property that for every vertex  $v \in V$  with  $f(v) = 0$ , either  $v$  is adjacent to a vertex assigned 2 under  $f$ , or  $v$  is adjacent to least two vertices assigned 1 under  $f$ . In this paper, we classify all graphs with Roman  $\{2\}$ -domination number belonging to the set  $\{2, 3, 4, n - 2, n - 1, n\}$ . Furthermore, we obtain some results about Roman  $\{2\}$ -domination number of some graph operations.

## 1. INTRODUCTION

We study *Roman  $\{2\}$ -dominating* functions defined in [3]. We first present some necessary terminology and notation. Let  $G = (V, E)$  be a graph with vertex set  $V = V(G)$  and edge set  $E(G)$ . The *open neighborhood*  $N(v)$  of a vertex  $v$  consists of the vertices adjacent to  $v$ , and its *closed neighborhood* is  $N[v] = N(v) \cup \{v\}$ . The degree of  $v$  is the cardinality of its open neighborhood. Let  $\Delta(G)$  be the maximum degree of the graph  $G$ . If  $S$  is a subset of  $V$ , then  $N(S) = \bigcup_{x \in S} N(x)$ ,  $N[S] = \bigcup_{x \in S} N[x]$ , and the subgraph induced by  $S$  in  $G$  is denoted  $G[S]$ .

A *dominating set* of  $G$  is a subset  $S$  of  $V$  such that every vertex in  $V - S$  has at least one neighbor in  $S$ , in other words,  $N[S] = V$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ . By [6], a subset  $S \subseteq V$  is a 2-dominating set if every vertex of  $V - S$  has at least two neighbors in  $S$ . The *2-domination number*  $\gamma_2(G)$  is the minimum cardinality of a 2-dominating set of  $G$ .

Motivated by Stewart's [10] article on defending the Roman Empire, Cockayne et al. introduced Roman dominating functions in [4]. For Roman domination, each vertex in the graph model corresponds to a location in the Roman Empire, and for protection, legions (armies) are stationed at various locations. A location is protected by a legion stationed there. A location having no legion can be protected by a legion sent from a neighboring location. However, this presents the problem of leaving a location unprotected (without a legion) when its legion is dispatched to a neighboring location. In order to prevent such problems, Emperor Constantine the Great [4] decreed that a legion cannot be sent to a neighboring location if it leaves its original station unprotected. In other words, every location with no legion must be adjacent to

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2010 *Mathematics Subject Classification.* Primary:05C69; Secondary: 05C76.

*Key words and phrases.* Roman  $\{2\}$ -domination; Cartesian product; Grid graph.

a location that has at least two legions. This defense strategy prompted the following definition in [4].

A function  $f : V(G) \rightarrow \{0, 1, 2\}$  is a Roman dominating function (RDF) on  $G$  if every vertex  $u \in V$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for which  $f(v) = 2$ . The weight of an RDF is the value  $f(V(G)) = \sum_{u \in V(G)} f(u)$ . The *Roman domination number*  $\gamma_R(G)$  is the minimum weight of an RDF on  $G$ . A vertex  $v$  with  $f(v) = 0$  is said to be undefended with respect to  $f$  if it is not adjacent to a vertex  $w$  with  $f(w) > 0$ .

In this paper, we study Roman  $\{2\}$ -dominating functions. These functions are closely related to  $\{2\}$ -dominating functions introduced in [5] as follows. For a graph  $G$ , a  $\{2\}$ -dominating function is a function  $f : V \rightarrow \{0, 1, 2\}$  having the property that for every vertex  $u \in V$ ,  $f(N[u]) \geq 2$ . The weight of a  $\{2\}$ -dominating function is the sum  $f(V) = \sum_{v \in V} f(v)$ , and the minimum weight of a  $\{2\}$ -dominating function  $f$  is the  $\{2\}$ -domination number, denoted by  $\gamma_{\{2\}}(G)$ .

A Roman  $\{2\}$ -dominating function  $f$  relaxes the restriction that for every vertex  $u \in V$ ,  $f(N[u]) = \sum_{v \in N[u]} f(v) \geq 2$  to only requiring that this property holds for every vertex assigned 0 under  $f$ . Formally, a Roman  $\{2\}$ -dominating function  $f : V \rightarrow \{0, 1, 2\}$  has the property that for every vertex  $v \in V$  with  $f(v) = 0$ ,  $f(N(u)) \geq 2$ , that is, either there is a vertex  $u \in N(v)$ , with  $f(u) = 2$ , or at least two vertices  $x, y \in N(u)$  with  $f(x) = f(y) = 1$ . In terms of the Roman Empire, this defense strategy requires that every location with no legion has a neighboring location with two legions, or at least two neighboring locations with one legion each. Note that for a Roman  $\{2\}$ -dominating function  $f$ , it is possible that  $f(N[v]) = 1$  for some vertex with  $f(v) = 1$ . The weight of a Roman  $\{2\}$ -dominating function is the sum  $f(V) = \sum_{v \in V} f(v)$ , and the minimum weight of a Roman  $\{2\}$ -dominating function  $f$  is the Roman  $\{2\}$ -domination number, denoted  $\gamma_{\{R2\}}(G)$ .

**Lemma 1.1.** [3, Corollary 10] *for a cycle  $C_n$  and a path  $P_n$  we have*

$$\gamma_{\{R2\}}(C_n) = \lceil \frac{n}{2} \rceil, \quad \gamma_{\{R2\}}(P_n) = \lceil \frac{n+1}{2} \rceil.$$

**Proposition 1.2.** [3, Proposition 5] *For every graph  $G$ ;  $\gamma_{\{R2\}}(G) \leq \gamma_2(G)$ .*

For graphs  $G$  and  $H$ , The *join* of graphs  $G$  and  $H$  is the graph  $G \vee H$  with the vertex set  $V = V(G) \cup V(H)$  where two vertices  $u$  and  $v$  are adjacent if

- ▷  $u, v \in V(G)$  and  $uv \in E(G)$  or
- ▷  $u, v \in V(H)$  and  $uv \in E(H)$  or
- ▷  $u \in V(G)$  and  $v \in V(H)$ .

The *Corona*  $G[H]$  of  $G$  and  $H$  is constructed as follows:

Choose a labeling of the vertices of  $G$  with labels  $1, 2, \dots, n$ . Take one copy of  $G$  and  $n$  disjoint copies of  $H$ , labeled  $H_1, \dots, H_n$ , and connect each vertex of  $H_i$  to vertex  $i$  of  $G$ .

The *Cartesian product* of two graphs  $G$  and  $H$ , denoted by  $G \square H$ , has vertex set  $V(G \square H) = V(G) \times V(H)$ , where two distinct vertices  $(u, v)$  and  $(x, y)$  of  $G \square H$  are

adjacent if either

$$u = x \text{ and } vy \in E(H) \text{ or } v = y \text{ and } ux \in E(G).$$

The *grid* graph  $G_{m,n}$  is the Cartesian product of  $P_m$  and  $P_n$ . In 1983, Jacobson and Kinch [9] established the exact values of  $\gamma(G_{m,n})$  for  $2 \leq m \leq 4$  which are the first results on the domination number of grids. Also, In 1993, Chang and Clark [2] found those of  $\gamma(G_{m,n})$  for  $m = 5$  and 6. Fischer found those of  $\gamma(G_{m,n})$  for  $m \leq 21$  (see Goncalves et al. [7]). Recently, Goncalves et al. [7] finished the computation of  $\gamma(G_{m,n})$  when  $24 \leq m \leq n$ . In [11], the authors have obtained the values of  $\gamma_2(G_{m,n})$  for  $2 \leq m \leq 4$ . In this paper, we will give some boundaries for  $\gamma_{\{R2\}}(G_{m,n})$  for  $2 \leq m \leq 4$ .

## 2. GRAPHS WITH SMALL OR LARGE ROMAN $\{2\}$ -DOMINATION NUMBER

In this section we provide a characterization of all connected graphs  $G$  of order  $n$  with Roman  $\{2\}$ -domination number belonging to  $\{2, 3, 4, n-2, n-1, n\}$ .

**Proposition 2.1.** *Let  $G$  be a graph.  $\gamma_{\{R2\}}(G) = 2$  if and only if  $G = \overline{K_n} \vee H$  for  $n = 1, 2$  and for some graph  $H$ .*

*Proof.* Let  $f = (V_0, V_1, V_2)$  be a  $\gamma_{\{R2\}}(G)$ -function with weight 2. Hence, we have two cases. If there exists a vertex  $z$  with  $z \in V_2$ , then all other vertices of  $G$  are adjacent to  $z$ . Therefore,  $G = K_1 \vee H$  for some graph  $H$ . If there are two vertices  $u$  and  $v$  are in  $V_1$ , then all other vertices of  $G$  are adjacent to both vertices  $u$  and  $v$ . If  $u$  and  $v$  are adjacent, then  $G = K_1 \vee H$  for some graph  $H$ , and if  $u$  and  $v$  are not adjacent, then  $G = \overline{K_2} \vee H$  for some induced subgraph  $H$  of  $G$ . Conversely, it is not hard to see the result.  $\square$

For a graph  $G$ , define  $N_i(G)$  for  $i = 1, \dots, n-1$  as follows,

$$N_i(G) = \{v \in V : \deg(v) = i\}.$$

**Proposition 2.2.** *Let  $G$  be a graph.  $\gamma_{\{R2\}}(G) = 3$  if and only if one of the following holds:*

- (i)  $\Delta(G) = n-2$  and  $N_{n-2}(G)$  is a clique,
- (ii)  $\Delta(G) < n-2$  and  $\gamma_2(G) = 3$ .

*Proof.* Let  $f = (V_0, V_1, V_2)$  be a  $\gamma_{\{R2\}}(G)$ -function with weight 3. By Proposition 2.1,  $\Delta(G) \leq n-2$ . At first, suppose that  $\Delta(G) = n-2$ . We consider two vertices  $u$  and  $v$  in  $N_{n-2}(G)$ . If  $u$  and  $v$  are not adjacent, then  $u$  and  $v$  are adjacent to all other vertices of  $G$ , and hence  $G = \overline{K_2} \vee H$ , which is a contradiction by Proposition 2.1. Thus,  $N_{n-2}(G)$  is a clique.

If  $\Delta(G) < n-2$ , then there are three vertices  $u, v$  and  $w$  in  $V_1$ . Hence,  $f$  is a 2-dominating function on  $G$ , and then  $\gamma_2(G) \leq 3$ . Since  $\gamma_{\{R2\}}(G) = 3$ , we have  $\gamma_2(G) \geq 3$ . So,  $\gamma_2(G) = 3$ . Moreover, the converse proof can be easily checked.  $\square$

**Proposition 2.3.** *Let  $G$  be a graph.  $\gamma_{\{R2\}}(G) = 4$  if and only if  $\Delta(G) \leq n-3$  and  $\gamma_2(G) \geq 4$  as well as  $G$  satisfies one of the following conditions,*

- (i)  $\gamma(G) = 2$ ,
- (ii)  $\gamma_2(G) = 4$ ,
- (iii) *There exists a vertex  $v \in V(G)$  such that  $\gamma_2(G[V(G) - N[v]]) = 2$ .*

*Proof.* Suppose that  $\gamma_{\{R2\}}(G) = 4$ . By Propositions 2.1 and 2.2, we have  $\Delta(G) \leq n-3$  and  $\gamma_{\{R2\}}(G) \geq 4$ . Let  $f = (V_0, V_1, V_2)$  be a  $\gamma_{\{R2\}}(G)$ -function. We consider three cases. First case, if  $|V_2| = 2$ , then  $\gamma(G) = 2$ . Second case,  $|V_1| = 4$ , so  $\gamma_2(G) = 4$ . Finally,  $|V_1| = 2$  and  $|V_2| = 1$ . Suppose that  $V_1 = \{u, w\}$  and  $V_2 = \{v\}$ . Obviously, each vertex in  $(V(G) - \{u, w\}) - N[v]$  must be connected to both  $u$  and  $w$ . Hence,  $\gamma_2(G[V(G) - N[v]]) = 2$ . Conversely, the result is obvious if we have (i) or (ii). Now, suppose that  $G$  satisfies (iii). Since  $\Delta(G) \leq n-3$  and  $\gamma_2(G) \geq 4$ , by Propositions 2.1 and 2.2,  $\gamma_{\{R2\}}(G) \geq 4$ . On the other hand, assume that  $\{u, w\}$  is a 2-dominating set for  $G[V(G) - N[v]]$ . If we assign a 2 to  $v$  and a 1 to  $u$  and  $w$ , we can show that  $\gamma_{\{R2\}}(G) \leq 4$ . Thus,  $\gamma_{\{R2\}}(G) = 4$ .  $\square$

**Proposition 2.4.** *Let  $G$  be a connected graph with order  $n$ . The following conditions are true,*

- (a)  $\gamma_{\{R2\}}(G) = n$  if and only if  $G = K_n$  for  $n = 1, 2$ .
- (b)  $\gamma_{\{R2\}}(G) = n-1$  if and only if  $G$  is a  $C_3$ ,  $P_3$  or  $P_4$ .

*Proof.* For (a) it is clear that  $\Delta(G) \leq 1$ . For (b), if  $G$  is one of the  $C_3$ ,  $P_3$  or  $P_4$ , then the claim is true. Conversely, assume that  $\gamma_{\{R2\}}(G) = n-1$ . Obviously  $\Delta(G) = 2$ . Among all  $\gamma_{\{R2\}}(G)$ -functions, let  $f = (V_0, V_1, V_2)$  be one with  $|V_2|$  as small as possible. It is easy to see that  $V_2 = \emptyset$  and  $|V_0| = 1$ . Suppose that  $v \in V_0$  for some vertex  $v \in V(G)$ , so  $\deg(v) = 2$ . Also, each vertex except  $v$  can be adjacent to at most one vertex in  $V_1$ . Hence, the vertices which have the degree 2 are at most  $v$  and  $N(v)$ . Therefore, we have just three graphs,  $C_3$ ,  $P_3$  or  $P_4$ .  $\square$

Now, we need the following graphs in Proposition 2.5.  $\hat{E}_6$  is a tree obtained from  $K_{1,3}$  by subdividing each edge exactly once.  $D_7$  is also a tree obtained from  $K_{1,3}$  by subdividing one edge three times, (see [1]). We define the graph  $H_2$  such that it is a graph with a 4-cycle and a path of order 2 joined to one of the vertices of the 4-cycle.

**Proposition 2.5.** *Let  $G$  be a connected graph with order  $n$ . Then  $\gamma_{\{R2\}}(G) = n-2$  if and only if  $G$  is one of the figures listed in Figure 1.*

*Proof.* Suppose that  $\gamma_{\{R2\}}(G) = n-2$ , then the following conditions hold,

- (i)  $\Delta(G) \leq 3$ ,
- (ii) each non-adjacent pair of vertices with degree 3 has exactly two common neighbours,
- (iii)  $G$  does not have one of the graphs  $P_7$ ,  $C_6$ ,  $\hat{E}_6$ ,  $D_7$ , and  $H_2$  as subgraph.

If there exists a vertex  $v \in V(G)$  with degree at least 4, then  $\gamma_{\{R2\}}(G) \leq n-3$ . Also, if there exists a pair of nonadjacent vertices with degree 3 having zero, one or three common neighbours, then we obtain  $\gamma_{\{R2\}}(G) \leq n-3$ . Moreover, Roman  $\{2\}$ -domination number of each of graphs  $P_7$ ,  $C_6$ ,  $\hat{E}_6$ ,  $D_7$ , and  $H_2$  is  $n-3$ . Thus, they

cannot be as a subgraph of  $G$ . It is not hard to see that all graphs which have the above three properties are listed in Figure 1. Conversely, it is easy to verify that for all graphs  $G$  listed in Figure 1, we have  $\gamma_{\{R2\}}(G) = n - 2$ .  $\square$

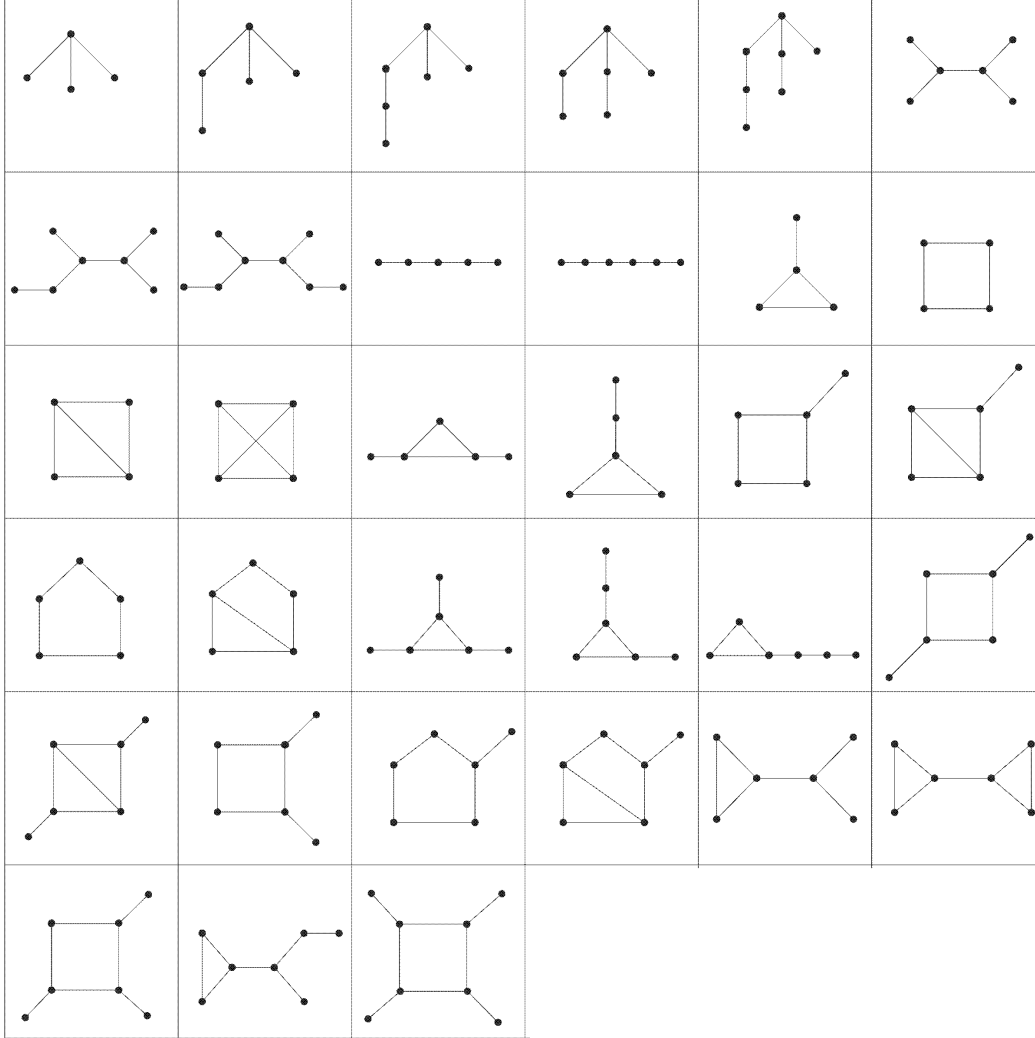


FIGURE 1.

### 3. GRAPH PRODUCTS

In this section we study Roman  $\{2\}$ -domination on some graph products. Also, in the following theorems we classify Roman  $\{2\}$ -domination for join of two graphs.

**Theorem 3.1.** *Let  $G$  and  $H$  be two graphs. Then  $\gamma_{\{R2\}}(G \vee H) \leq 4$ . Moreover, if  $k = \gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(H)$ , then we have*

- (a)  $k \leq 2$  if and only if  $\gamma_{\{R2\}}(G \vee H) = 2$ ,  
 (b)  $k = 3$  or  $k = 4$  and  $\gamma(G) = 2$  if and only if  $\gamma_{\{R2\}}(G \vee H) = 3$ .

*Proof.* The first assertion is obvious because for each graph  $G$ ,  $\gamma_{\{R2\}}(G) \leq 2\gamma(G)$ . For (a), assume that  $k = 1$ , then  $G = K_1$ . It is sufficient to use Proposition 2.1. Now, suppose  $k = 2$ . By Proposition 2.1,  $G \vee H = \overline{K_n} \vee F$  for  $n = 1, 2$  and for some graph  $F$ . Conversely, let  $\gamma_{\{R2\}}(G \vee H) = 2$ . By Proposition 2.1, there exists a graph  $L$  such that  $G \vee H = \overline{K_n} \vee L$  for  $n = 1, 2$ . It is not hard to see that the vertices of  $\overline{K_n}$  for  $n = 1, 2$  together belong to  $G$  or  $H$ . Anyway,  $\gamma_{\{R2\}}(G) \leq 2$ .

For (b), if  $k = 3$ , then  $\gamma_{\{R2\}}(G \vee H) \leq 3$ . By (a),  $\gamma_{\{R2\}}(G \vee H) \geq 3$ . For the second claim, let  $\{u, v\} \subseteq V(G)$  be a minimum dominating set for  $G$  and  $w$  be an arbitrary vertex in  $V(H)$ . It is seen that  $\{u, v, w\}$  is a 2-dominating set for  $G \vee H$ . Using Proposition 2.2 we have  $\gamma_{\{R2\}}(G \vee H) = 3$ . Conversely, let  $\gamma_{\{R2\}}(G \vee H) = 3$ . By (a),  $k \geq 3$ . First assume that  $\gamma_2(G \vee H) = 3$ . Let  $\{u, v, w\} \subseteq V(G \vee H)$  be a 2-dominating set on  $G \vee H$ . Without loss of generality, we consider two subcases,

- (i) If  $\{u, v, w\} \subseteq V(G)$ , then by (a),  $\gamma_{\{R2\}}(G) = 3$ .  
 (ii) If  $\{u, v\} \subseteq V(G)$  and  $w \in V(H)$ , then  $\gamma(G) = 2$ . So by (a),  $3 \leq \gamma_{\{R2\}}(G) \leq 4$ .

For the next case, there exist two vertices  $u, v \in V(G \vee H)$  with label 1 and 2, respectively. It is not hard to see that  $u, v \in V(G)$  or  $u, v \in V(H)$ . Therefore,  $k = 3$ .  $\square$

In the following theorem we obtain Roman  $\{2\}$ -domination number for the Corona product of two graphs.

**Theorem 3.2.** *Let  $G$  and  $H$  be two graphs such that the order of  $G$  is  $n$ . If  $H = K_1$ , then  $\gamma_{\{R2\}}(G[H]) = n + \gamma(G)$ , otherwise  $\gamma_{\{R2\}}(G[H]) = 2n$ .*

*Proof.* Let  $H = K_1$ . Easily we can show that for every graph  $G$ ,  $\gamma_{\{R2\}}(G[K_1]) \leq n + \gamma(G)$ . On the other hand, assume that  $f = (V_0, V_1, V_2)$  is a  $\gamma_{\{R2\}}(G[K_1])$ -function. Without loss of generality, suppose that  $nK_1 \subseteq V_0 \cup V_1$ . Also, let  $\ell K_1 \in V_1$  and  $(n - \ell)K_1 \in V_0$ . Thus,  $V_2 \subseteq V(G)$ . Moreover,  $(V_1 \cap V(G)) \cup V_2$  forms a dominating set for  $G$ .

$$\begin{aligned} wt(f) &= \ell + |V_1 \cap V(G)| + 2|V_2| \\ &\geq \ell + \gamma(G) + |V_2| \\ &\geq n + \gamma(G). \end{aligned}$$

For the second assertion, let  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $f$  be a  $\gamma_{\{R2\}}(G[H])$ -function. Then,

$$wt(f) = wt(f|_{H_1}) + wt(f|_{H_2}) + \dots + wt(f|_{H_n}) \geq 2n,$$

where  $H_i = v_i \vee H$  for  $i = 1, 2, \dots, n$ . So,  $\gamma_{\{R2\}}(G[H]) = 2n$ .  $\square$

Moreover, we state a bound and some results about Cartesian product of graphs. Let  $G$  and  $H$  be two graphs with  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $V(H) = \{u_1, u_2, \dots, u_m\}$ .

In  $G \square H$ , we define  $G^i$  and  $H^j$  for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , as  $i$ th layer and  $j$ th layer of  $G$  and  $H$ , respectively as follows,

$$G^i = \{(v, u_i) : v \in V(G)\}, \quad H^j = \{(v_j, u) : u \in V(H)\}.$$

**Theorem 3.3.**  $\gamma_{\{R2\}}(G \square H) \leq \min \{\gamma_{\{R2\}}(G)|V(H)|, \gamma_{\{R2\}}(H)|V(G)|\}$ . Also, this bound is sharp.

*Proof.* Let  $f$  be a  $\gamma_{\{R2\}}$ -function for  $H$ . Consider each copy of  $H$  with  $\gamma_{\{R2\}}$ -function  $f$  in cartesian product  $G \square H$ . Since we have  $|V(G)|$  copies of  $H$ , it is easy to see that  $\gamma_{\{R2\}}(G \square H) \leq \gamma_{\{R2\}}(H)|V(G)|$ . By a similar way, we have  $\gamma_{\{R2\}}(G \square H) \leq \gamma_{\{R2\}}(G)|V(H)|$ . In order to prove this bound is sharp, consider  $\gamma_{\{R2\}}(K_{1,n} \square P_2) = 2\gamma_{\{R2\}}(K_{1,n}) = 4$ , for  $n \geq 3$ , see Proposition 2.3.  $\square$

**Theorem 3.4.** Let  $m$  and  $n$  be two positive integers with  $n \leq m$ . Then

$$\gamma_{\{R2\}}(K_n \square K_m) = \min\{m, 2n\}.$$

*Proof.* Let  $V(K_n) = \{v_1, v_2, \dots, v_n\}$  and  $V(K_m) = \{u_1, u_2, \dots, u_m\}$ . Suppose that  $\gamma_{\{R2\}}(K_n \square K_m) < \min\{m, 2n\}$ , and let  $f = (V_0, V_1, V_2)$  be a  $\gamma_{\{R2\}}(K_n \square K_m)$ -function. Thus, we can say that there exists the layer  $K_n^i$  for some  $1 \leq i \leq m$ , such that  $wt(f|_{K_n^i}) = 0$ . On the other hand, we can find a layer  $K_m^j$  for some  $1 \leq j \leq n$ , with  $wt(f|_{K_m^j}) \leq 1$ . It is easy to see that  $(v_i, u_j) \in V_0$  and  $f(N(v_i, u_j)) \leq 1$ . Therefore, we achieve a contradict. Now to get the equality, consider a Roman  $\{2\}$ -dominating function on  $K_n \square K_m$  that assigns to  $(v_i, u_i)$  and  $(v_1, u_j)$  a 1 for every  $i$  and for every  $j$  belonging to  $\{n+1, \dots, m\}$ , and a 0 to the remaining vertices of the graph.  $\square$

We know that  $\gamma_{\{R2\}}(G_{m,n}) \leq \gamma_2(G_{m,n})$  for all positive integers  $m$  and  $n$ . Moreover, this bound is sharp for  $G_{2,n}$  for each  $n$  and  $G_{3,n}$  for  $n \leq 13$  as well as  $G_{4,4}$ . We recall the following results of [11].

**Theorem 3.5.** Let  $n$  be a positive integer. Then the following equalities hold:

- (i)  $\gamma_2(G_{2,n}) = n$ ,
- (ii)  $\gamma_2(G_{3,n}) = \lceil \frac{4n}{3} \rceil$ ,
- (iii)  $\gamma_2(G_{4,n}) = \lceil \frac{7n+3}{4} \rceil$ , for  $n \geq 3$ .

**Proposition 3.6.**  $\gamma_{\{R2\}}(G_{2,n}) = n$ .

*Proof.* We claim that the weight of each layer of  $P_2$  is at least 1. Assume that there exists a layer with weight 0. To have a Roman  $\{2\}$ -dominating set for  $G_{2,n}$ , the weight of the adjacent layers will be 4. The obtained Roman  $\{2\}$ -domination number is not optimal because its weight is larger than  $\gamma_2(G_{2,n})$ .  $\square$

**Proposition 3.7.**

- (a) For  $n = 2, 3, 6$ ,  $\gamma_{\{R2\}}(G_{3,n}) \leq \lfloor \frac{5n+3}{4} \rfloor$ . Otherwise,  $\gamma_{\{R2\}}(G_{3,n}) \leq \lceil \frac{5n+3}{4} \rceil$ .
- (b) For  $n = 2, 3, 5, 6, 9$ ,  $\gamma_{\{R2\}}(G_{4,n}) \leq \lfloor \frac{5n+4}{3} \rfloor$ . Otherwise,  $\gamma_{\{R2\}}(G_{4,n}) \leq \lceil \frac{5n+4}{3} \rceil$ .

*Proof.* Suppose that  $v_{ij}$  is the vertex in the row  $i$  and column  $j$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$  in  $G_{m,n}$ . In each part we give a complete explanation about a basic case of the product and then we can obtain upper cases using it. For (a), let  $n = 4k - 1$  for some positive integer  $k \geq 2$ . We define a Roman  $\{2\}$ -dominating function  $f = (V_0, V_1, V_2)$  such that  $v_{ij} \in V_2$  for  $j = 4t$  for some positive integer  $1 \leq t \leq k - 1$ , such that  $i = 1$  if  $t$  is odd, otherwise  $i = 3$ . Also,

$$V_0 = \{v_{ij} : d(v_{ij}, v) = 1, 2, 4, \text{ for some } v \in V_2, \text{ and } 1 \leq i \leq 3, 1 \leq j \leq n\},$$

where  $d(v_{ij}, v)$  is the length of shortest path between two vertices  $v_{ij}$  and  $v$ . The label of other vertices is 1. Hence,  $wt(f) = 5k$ . For  $n \neq 4k - 1$  we obtain the result by adding at most 3 columns to the case  $n = 4k - 1$ . For (b), in figures A, B and C in Figure 2, a star, a black circle and a white circle denote a vertex with label 2, 1 and 0, respectively. We want to construct  $G_{4,n}$  for  $n \geq 7$  by merging a number of figures A, B and C. Suppose that  $n(A), n(B)$  and  $n(C)$  are the number of used  $A, B$  and  $C$  in  $G_{4,n}$ , respectively. Consider  $n = 3k + i$  for some positive integers  $k$  and  $i$  such that  $1 \leq i \leq 3$ . For  $G_{4,n}$  assign  $n(A) = k - i, n(C) = i - 1$  and  $n(B) = 1$  except for  $n = 9$ ,  $n(B) = 0$ .  $\square$

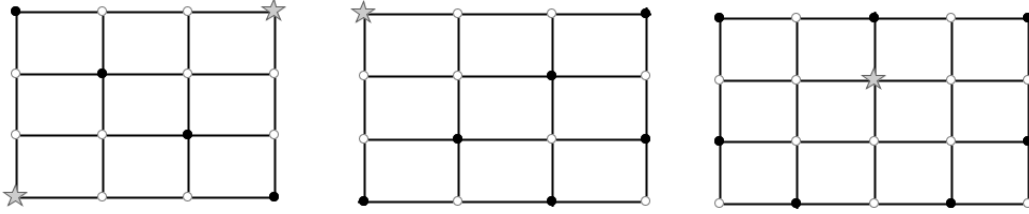


FIGURE 2. A, B and C, respectively

#### ACKNOWLEDGMENT

We would like to thank Mustapha Chellali for his useful comments.

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